Tutorial 1

Microeconomics 3, Part 2

Question 1

Consider a pure exchange economy with $I = 3$ consumers and $L = 3$ goods with the following utility functions and initial endowments:

$$
\begin{aligned}
u_1(x_{11}, x_{21}, x_{31}) &= \min\{x_{11}, x_{21}\} \qquad \text{with} \qquad \boldsymbol{\omega}_1 = (\omega_{11}, 0, 0)\\
u_2(x_{12}, x_{22}, x_{32}) &= \min\{x_{22}, x_{32}\} \qquad \text{with} \qquad \boldsymbol{\omega}_2 = (0, \omega_{22}, 0)\\
u_3(x_{13}, x_{23}, x_{33}) &= \min\{x_{13}, x_{33}\} \qquad \text{with} \qquad \boldsymbol{\omega}_3 = (0, 0, \omega_{33})
\end{aligned}
$$

The consumption set, X_i, is $\mathbb{R}^3_+$ for all consumers. There is a single firm whose only production technology is free disposal ($Y_1 = -\mathbb{R}^3_+$). Each consumer owns a one-third share in the profits of this firm ($\theta_{i1} = \frac{1}{3}$ for all $i = 1, 2, 3$).

(i) Suppose $\omega_{11} = 1$, $\omega_{22} = \frac{1}{2}$ and $\omega_{33} = 1$. Solve for a Walrasian equilibrium in this economy and provide the normalized price vector and equilibrium allocation for each consumer and the firm. You may normalize the price of good 1 to 1.

(ii) Suppose instead $\omega_{11} = 1$, $\omega_{22} = 2$ and $\omega_{33} = 1$. Is there a Walrasian equilibrium in this economy? If so, solve for the price vector and allocation as in (i). If not, show that there cannot be one.

Solution:

Because of the pure complements utility function for each consumer, a consumer i will demand $\frac{\boldsymbol{p}\cdot\boldsymbol{\omega}_i}{p_\ell + p_{\ell'}}$ if goods ℓ and ℓ' are in their utility function and zero otherwise.

(i) In this case the excess demand functions for each consumer is:

$$
\boldsymbol{z}_1(\boldsymbol{p}) = \begin{pmatrix} \frac{p_1}{p_1+p_2} - 1 \\ \frac{p_1}{p_1+p_2} \\ 0 \end{pmatrix} \qquad \boldsymbol{z}_2(\boldsymbol{p}) = \begin{pmatrix} 0 \\ \frac{p_2}{2(p_2+p_3)} - \frac{1}{2} \\ \frac{p_2}{2(p_2+p_3)} \end{pmatrix} \qquad \boldsymbol{z}_3(\boldsymbol{p}) = \begin{pmatrix} \frac{p_3}{p_1+p_3} \\ 0 \\ \frac{p_3}{p_1+p_3} - 1 \end{pmatrix}
$$

The excess demand function is then:

$$\boldsymbol{z}(\boldsymbol{p}) = \begin{pmatrix} \frac{p_1}{p_1+p_2} + \frac{p_3}{p_1+p_3} - 1 \\ \frac{p_1}{p_1+p_2} + \frac{p_2}{2(p_2+p_3)} - \frac{1}{2} \\ \frac{p_2}{2(p_2+p_3)} + \frac{p_3}{p_1+p_3} - 1 \end{pmatrix}$$

We search for a price vector to make $\boldsymbol{z}(\boldsymbol{p}) = \mathbf{0}$, which is the case when $\boldsymbol{p} \gg \mathbf{0}$ and there is no firm destruction. From the equation for the first good (with the $p_1 = 1$ normalization) we can get:

$$\begin{aligned} \frac{1}{1+p_2} + \frac{p_3}{1+p_3} &= 1 \\ (1+p_3) + p_3(1+p_2) &= (1+p_2)(1+p_3) \\ 1 + p_3 + p_3 + p_2 p_3 &= 1 + p_2 + p_3 + p_2 p_3 \\ p_3 &= p_2 \end{aligned}$$

Using this in equation 2:

$$\begin{aligned} \frac{1}{1+p_2} + \frac{p_2}{2(p_2+p_2)} &= \frac{1}{2} \\ \frac{1}{1+p_2} + \frac{1}{4} &= \frac{1}{2} \\ 1 + p_2 &= 4 \\ p_2 &= 3 \end{aligned}$$

So $p_1 = 1$ and $p_2 = p_3 = 3$ in equilibrium. The allocation is:

$$\begin{aligned} \boldsymbol{x}_1 &= \left(\frac{1}{4}, \frac{1}{4}, 0\right) \\ \boldsymbol{x}_2 &= \left(0, \frac{1}{4}, \frac{1}{4}\right) \\ \boldsymbol{x}_3 &= \left(\frac{3}{4}, 0, \frac{3}{4}\right) \\ \boldsymbol{y}_1 &= (0, 0, 0) \end{aligned}$$

(ii) In this case the excess demand functions for each consumer is:

$$\boldsymbol{z}_1(\boldsymbol{p}) = \begin{pmatrix} \frac{p_1}{p_1+p_2} - 1 \\ \frac{p_1}{p_1+p_2} \\ 0 \end{pmatrix} \qquad \boldsymbol{z}_2(\boldsymbol{p}) = \begin{pmatrix} 0 \\ \frac{2p_2}{p_2+p_3} - 2 \\ \frac{2p_2}{p_2+p_3} \end{pmatrix} \qquad \boldsymbol{z}_3(\boldsymbol{p}) = \begin{pmatrix} \frac{p_3}{p_1+p_3} \\ 0 \\ \frac{p_3}{p_1+p_3} - 1 \end{pmatrix}$$

The excess demand function is then:

$$\boldsymbol{z}(\boldsymbol{p}) = \begin{pmatrix} \frac{p_1}{p_1+p_2} + \frac{p_3}{p_1+p_3} - 1 \\ \frac{p_1}{p_1+p_2} + \frac{2p_2}{p_2+p_3} - 2 \\ \frac{2p_2}{p_2+p_3} + \frac{p_3}{p_1+p_3} - 1 \end{pmatrix}$$

We search for a price vector to make $\boldsymbol{z}(\boldsymbol{p}) = \mathbf{0}$ (which is the case when $\boldsymbol{p} \gg \mathbf{0}$ and there is no firm disposal). From the equation for the first good we can get (this is the same as before):

$$\begin{aligned} \frac{1}{1+p_2} + \frac{p_3}{1+p_3} &= 1 \\ (1+p_3) + p_3(1+p_2) &= (1+p_2)(1+p_3) \\ 1 + p_3 + p_3 + p_2p_3 &= 1 + p_2 + p_3 + p_2p_3 \\ p_3 &= p_2 \end{aligned}$$

Using this in equation 2:

$$\begin{aligned} \frac{1}{1+p_2} + \frac{2p_2}{p_2+p_3} &= 2 \\ \frac{1}{1+p_2} + 1 &= 2 \\ 1 + p_2 &= 1 \\ p_2 &= 0 \end{aligned}$$

This implies that $p_1 = 1$ and $p_2 = p_3 = 0$. However, if $p_2 = p_3 = 0$, then consumer 2 will demand an infinite amount of both goods 2 and 3 because they are both free (despite having a value of wealth equal to zero). Therefore this cannot be an equilibrium.

We now consider the possibility of finding a price vector such that $\boldsymbol{z}(\boldsymbol{p}) \leq \mathbf{0}$ with $\boldsymbol{z}(\boldsymbol{p}) \neq \mathbf{0}$, i.e. where the firm destroys at least one of the goods. As we saw above, if two goods have a zero price, then one of the consumers will demand an infinite amount of one of the goods. If all 3 have a zero price, then all consumers will demand an infinite amount of all goods. Therefore the only possibility left is that only one price is zero and the others are positive.

If, with the normalization $p_1 = 1$, then as we saw above, no matter what we will have $p_2 = p_3$ as the firm will not want to destroy good 1 when it has a positive price. Therefore it is not possible for only one of goods 2 or 3 to have a price equal to zero. Suppose we consider alternative normalizations to see if we can have $p_1 = 0$ with $p_2 > 0$ and $p_3 > 0$. If we normalized $p_2 = 1$, then from the equation for the first good, we have:

$$\sum_{i=1}^{3} x_{1i}\left(\boldsymbol{p}, \boldsymbol{p} \cdot \boldsymbol{\omega}_i\right) = \bar{\omega}_1 + y_{11}$$
$$\frac{0}{0+1} + \frac{p_3}{0+p_3} = 1 + y_{11}$$

From this we can see that for any p_3 we have $y_{11} = 0$, so no destruction. Suppose instead $p_3 = 1$:

$$\frac{0}{0+p_2} + \frac{1}{0+1} = 1 + y_{11}$$

Again, for any p_2 we will have $y_{11} = 0$. Therefore there cannot be an equilibrium with this endowment.

Question 2

There is an economy with two goods, two consumers and one firm. One of the goods (good x) is a public good and the other good (good m) is a rival and excludable composite commodity. Each consumer's utility function is:

$$u_1\left(x, m_1\right) = \log\left(x\right) + m_1$$
$$u_2\left(x, m_2\right) = 2\log\left(x\right) + m_2$$

The firm's production technology is as follows: one unit of good x can be produced from two units of good m. Each consumer is endowed with 0 units of good x and 5 units of good m, so $\boldsymbol{\omega}_i = (0, 5)$ for $i = 1, 2$. Each consumer has an equal share in the profits of the firm. Normalize the price of good m to 1 and use p for the price of good x.

(i) Show that the Pareto optimal quantity of the public good is $x^{\circ} = \frac{3}{2}$.

(ii) What is the price p and allocation $(x^{\star}, m_1^{\star}, m_2^{\star})$ in the competitive equilibrium of this economy?

(iii) If a per-unit subsidy is placed on good x, what does the subsidy need to be in order to achieve the Pareto optimal quantity, x°?

(iv) Suppose the subsidy you found in (iii) needs to be financed by a per-unit tax, t, on the consumption of good m. What should the tax on good m be to balance the government's budget (i.e. the total revenue from the tax equals the total cost of the subsidy)?

(v) *Note: This question is unrelated to (iii) and (iv).* Suppose the government taxed individuals based on "not contributing their fair share" of the public good. The tax paid by a consumer who contributes x_i to the public good is:

$$\begin{cases} t\left(\frac{x^{\circ}}{2} - x_i\right) & \text{if } x_i < \frac{x^{\circ}}{2} \\ 0 & \text{if } x_i \geq \frac{x^{\circ}}{2} \end{cases}$$

where $x^{\circ} = \frac{3}{2}$ and $t = \frac{3}{2}$. Consumer i is taxed if they contribute less than half of the Pareto optimal quantity, x°. However, if they contribute at least half, they are not taxed.

Show that with $t = \frac{3}{2}$ there is an equilibrium in which both consumers choose $x_i = \frac{x^{\circ}}{2} = \frac{3}{4}$, and the Pareto optimal quantity of the public good is provided.

Solution:

(i) Any units of good m not consumed is used to produce good x as utility is increasing in both goods. With this, the planner's problem is:

$$\max_{(x, m_1, m_2) \geq \mathbf{0}} \log(x) + m_1 + 2\log(x) + m_2 \quad \text{subject to} \quad x = \frac{1}{2}(10 - m_1 - m_2)$$

Substituting $m_1 + m_2 = 10 - 2x$ into the objective:

$$\max_{x \geq 0} 3\log(x) + 10 - 2x$$

The FOC is $\frac{3}{x} - 2 = 0$ ($x = 0$ cannot be a solution), so $x^\circ = \frac{3}{2}$.

(ii) Consumer i's problem is:

$$\max_{(x_i, m_i) \geq \mathbf{0}} i \log\left(x_i + x^\star_{-i}\right) + m_i \text{ subject to } px_i + m_i \leq 5$$

The budget constaint will hold with equality at the optimum since utility is increasing in both goods, so the problem becomes:

$$\max_{x_i \geq 0} i \log\left(x_i + x^\star_{-i}\right) + 5 - px_i$$

The FOC for each consumer is:

$$\frac{i}{x_i + x^\star_{-i}} - p \leq 0 \text{ with equality if } x_i > 0$$

Consider the following 4 cases:

- We cannot have $x_1 > 0$ and $x_2 > 0$ as both FOCs can't hold with equality simultaneously.
- If $x_2 = 0$, then we would have $x_1 = \frac{1}{p}$, but then consumer 1's FOC would be $\frac{1}{\frac{1}{p}+0} - p = 0$, which cannot satisfy the FOC unless $p = 0$. But in this case, an infinite amount of x would be demanded.
- If $x_1 = x_2 = 0$, then the inequality in both FOCs would be violated.
- If $x_1 = 0$, then $x_2 = \frac{2}{p}$. This satisfies 1's FOC: $\frac{1}{0+\frac{2}{p}} - p = -\frac{p}{2} < 0$.

The price of the good must be 2 in any equilibrium with a positive quantity of good x, since if $p > 2$, the firm would supply an infinite quantity, and if $p < 2$ the firm would supply zero. At $p = 2$ the firm makes zero profits with any quantity. So the equilibrium is $p = 2$, $x = 1$ and $m_1 = 5$ and $m_2 = 3$.

(iii) With a subsidy, the FOCs become:

$$\frac{i}{x_i + x^\star_{-i}} - p + s \leq 0 \text{ with equality if } x_i > 0$$

By the same argument as in (ii), $x_1 = 0$ and $x_2 > 0$. Similarly we need $p = 2$. So consumer 2's FOC is $\frac{2}{x_2} - 2 + s = 0$, so to achieve $x_2 = \frac{3}{2}$, we need $s = 2 - \frac{2}{\frac{3}{2}} = \frac{2}{3}$. The total cost of subsidy is then $\frac{3}{2} \times \frac{2}{3} = 1$.

(iv) The government needs to make revenue of 1 (the total cost of the subsidy). Both consumers spend their leftover income after their expenditure on good x on good m. So for consumer 1, who spends no income on good x, with a tax they will demand $m_1 = \frac{5}{1+t}$ of good m. For consumer 2, we saw above they buy $x_2 = \frac{3}{2}$ at an effective price of $p - s = 2 - \frac{2}{3} = \frac{4}{3}$. So:

$$m_2 = \frac{1}{1+t}\left[5 - (p-s)\, x_2\right] = \frac{1}{1+t}\left(5 - \frac{3}{2} \times \frac{4}{3}\right) = \frac{3}{1+t}$$

The aggregate demand for good m is then $\frac{8}{1+t}$. We need to find the tax t that makes the revenue exactly 1:

$$t \times \frac{8}{1+t} = 1 \quad \Longrightarrow \quad t = \frac{1}{7}$$

(v) Consumer i's problem is now

$$\max_{(x_i, m_i) \geq \mathbf{0}} i \log\left(x_i + x^\star_{-i}\right) + m_i \text{ subject to } px_i + t\left(\frac{3}{4} - x_i\right) \mathbb{1}\left\{x_i < \frac{3}{4}\right\} + m_i \leq 5$$

The budget constaint will hold with equality at the optimum since utility is increasing in both goods:

$$\max_{x_i \geq 0} i \log\left(x_i + x^\star_{-i}\right) + 5 - px_i - t\left(\frac{3}{4} - x_i\right) \mathbb{1}\left\{x_i < \frac{3}{4}\right\}$$

For the same reasons as above, we will have $p = 2$ in any equilibrium. If the other consumer chooses $x_{-i} = \frac{x^\circ}{2}$, the objective for consumer i is (using $t = \frac{3}{2}$

and omitting the constant 5):

$$\max_{x_i \geq 0} i \log\left(x_i + \frac{3}{4}\right) - 2x_i - \left(\frac{9}{8} - \frac{3}{2}x_i\right) \mathbb{1}\left\{x_i < \frac{3}{4}\right\}$$

- If consumer 2 chooses $x_2 = \frac{3}{4}$:
  - Suppose the optimal choice for consumer 1 was $x_1 \in \left(0, \frac{3}{4}\right)$. Then the FOC would be $\frac{1}{x_1 + \frac{3}{4}} - 2 + \frac{3}{2} = 0$. But $x_1 = \frac{5}{4}$ solves this equation, so we cannot have $x_1 \in \left(0, \frac{3}{4}\right)$.
  - Suppose the optimal choice for consumer 1 was $x_1 > \frac{3}{4}$. Then the FOC would be $\frac{1}{x_1 + \frac{3}{4}} - 2 = 0$. But $x_1 = -\frac{1}{4}$ solves this equation, so we cannot have $x_1 > \frac{3}{4}$.
  - If $x_1 = 0$, the objective is $\log\left(\frac{3}{4}\right) - \frac{9}{8}$.
  - If $x_1 = \frac{3}{4}$, the objective is $\log\left(\frac{3}{2}\right) - 2 \times \frac{3}{4}$
  - The optimal choice is then $x_1 = \frac{3}{4}$ since $\log\left(\frac{3}{2}\right) - 2 \times \frac{3}{4} > \log\left(\frac{3}{4}\right) - \frac{9}{8}$.
- If consumer 1 chooses $x_1 = \frac{3}{4}$:
  - Suppose the optimal choice for conumser 2 was $x_2 \in \left(0, \frac{3}{4}\right)$. Then the FOC would be $\frac{2}{x_1 + \frac{3}{4}} - 2 + \frac{3}{2} = 0$. But $x_2 = \frac{13}{4}$ solves this equation, so we cannot have $x_2 \in \left(0, \frac{3}{4}\right)$.
  - Suppose the optimal choice for consumer 2 was $x_2 > \frac{3}{4}$. Then the FOC would be $\frac{2}{x_2 + \frac{3}{4}} - 2 = 0$. But $x_2 = -\frac{3}{4}$ solves this equation, so we cannot have $x_2 > \frac{3}{4}$.
  - If $x_2 = 0$, the objective is $2\log\left(\frac{3}{4}\right) - \frac{9}{8}$.
  - If $x_2 = \frac{3}{4}$, the objective is $2\log\left(\frac{3}{2}\right) - 2 \times \frac{3}{4}$
  - The optimal choice is $x_2 = \frac{3}{4}$ since $2\log\left(\frac{3}{2}\right) - 2 \times \frac{3}{4} > 2\log\left(\frac{3}{4}\right) - \frac{9}{8}$.

Since when $p = 2$, $x_1 = x_2 = \frac{3}{4}$ is optimal for both consumers if the other consumer chooses $x_{-i} = \frac{3}{4}$, it is an equilibrium. Since $x_1 + x_2 = \frac{3}{2} = x^\circ$, the Pareto optimal quantity of the public good is provided.

Question 3

In a pure exchange economy there are two consumers and two goods. The two consumers are are two individuals who share an apartment. The two goods are money and cigarettes. Consumer 1 enjoys smoking cigarettes but Consumer 2 obtains disutility when Consumer 1 smokes.

Each consumer's utility is:

$$u_1(x_{11}, x_{21}) = \sqrt{x_{11}x_{21}}$$
$$u_2(x_{12}, x_{21}) = \sqrt{x_{12}(20 - x_{21})}$$

where $x_{\ell i}$ is i's consumption of the ℓth good, where good 1 is money and good 2 is cigarettes. Notice that if Consumer 1 consumes more cigarettes, Consumer 2's utility decreases. If Consumer 2 consumes cigarettes, neither consumer's utility is affected (think of Consumer 2 as freely destroying any cigarettes that Consumer 1 doesn't smoke).

The total endowment of money in the economy is €100 and the total endowment of cigarettes is 20. The endowment of money is split equally among consumers (€50 each). The distribution of the endowment of cigarettes can vary question-by-question.

In parts (i)-(ii), there is a market for both goods. In this market, the price per cigarette is p and the price of money is normalized to 1.

Hint: Since there are 20 cigarettes in total in the economy, you could consider Consumer 2's utility function as $u_2(x_{12}, x_{22}) = \sqrt{x_{12}x_{22}}$ where $x_{22} = 20 - x_{21}$ is the number of cigarettes *not smoked* by consumer 1.

(i) Suppose Consumer 1 owns the full endowment of cigarettes and there is a market for both goods. What is the equilibrium price and allocation? How many cigarettes are smoked in total?

(ii) Now suppose *Consumer 2* owns the full endowment of cigarettes. What is the equilibrium price per cigarette and allocation?

(iii) Find the full set of Pareto allocations for this economy.

(iv) Consumer 1 owns the full endowment of cigarettes like in part (i), but there is no longer a market for cigarettes. Show that if Consumer 2 can make a take-it-or-leave-it offer to Consumer 1 to reduce cigarette consumption, that the optimal offer is $T = 50\left(\sqrt{2} - 1\right) \approx$ €20.71 to reduce consumption to $\frac{20}{\sqrt{2}} \approx 14.14$ cigarettes.

Solution:

(i) Consumer 1's problem is:

$$\max_{(x_{11},x_{21})\geq\mathbf{0}} \sqrt{x_{11}x_{21}} \text{ subject to } x_{11}+px_{21}\leq\omega_{11}+p\omega_{21}$$

This is standard Cobb-Douglas demand, so

$$\boldsymbol{x}_1(p,\boldsymbol{\omega}_1)=\left(\frac{\omega_{11}+p\omega_{21}}{2},\frac{\omega_{11}+p\omega_{21}}{2p}\right)$$

For consumer 2's problem, define $x_{22}=20-x_{21}$, the amount of cigarettes *not smoked*. The problem is then:

$$\max_{(x_{12},x_{22})\geq\mathbf{0}} \sqrt{x_{12}x_{22}} \text{ subject to } x_{12}+px_{22}\leq\omega_{12}+p\omega_{22}$$

This is standard Cobb-Douglas demand, so

$$\boldsymbol{x}_2(p,\boldsymbol{\omega}_2)=\left(\frac{\omega_{12}+p\omega_{22}}{2},\frac{\omega_{12}+p\omega_{22}}{2p}\right)$$

Given the endowment for this question, consumer 1's demand is $\boldsymbol{x}_1=\left(25+10p,\frac{25}{p}+10\right)$ and consumer 2's demand is $\boldsymbol{x}_2=\left(25,\frac{25}{p}\right)$. For market clearing in cigarettes, we need

$$\underbrace{\frac{25}{p}+10}_{\text{smoked}}+\underbrace{\frac{25}{p}}_{\text{not smoked}}=20$$

A price of $p=5$ clears the market in this good. We can see that for good 1, the market also clears at this price: $25+10\times5+25=100$. Consumer 1 sells 5 cigarettes to Consumer 2 and consumes 15.

(ii) Given the endowment for this question, consumer 1's demand is $\boldsymbol{x}_1=\left(25,\frac{25}{p}\right)$ and consumer 2's demand is $\boldsymbol{x}_2=\left(25+10p,\frac{25}{p}+10\right)$. From market clearing, we can see the price is $p=5$ again, only this time consumer 1 consumes 5 cigarettes instead of 15. Consumer 1 buys 5 cigarettes from Consumer 2.

(iii) The planner's problem is:

$$\max_{(x_{11},x_{21},x_{12})\geq \mathbf{0}} \sqrt{x_{11}x_{21}} \qquad \text{subject to:} \qquad \sqrt{x_{12}\left(20-x_{21}\right)} \geq \underline{u}_2$$
$$\text{and} \qquad x_{11}+x_{12} \leq 100$$
$$\text{and} \qquad x_{21} \in [0,20]$$

Since utility for both consumers is increasing in money, it must be the case that in any Pareto optimal allocation that $x_{12} = 100 - x_{11}$. The problem is then:

$$\max_{(x_{11},x_{21})\geq \mathbf{0}} \sqrt{x_{11}x_{21}} \qquad \text{subject to:} \qquad \sqrt{\left(100-x_{11}\right)\left(20-x_{21}\right)} \geq \underline{u}_2$$
$$\text{and} \qquad x_{21} \in [0,20]$$

At an interior solution (each $x_{\ell i} > 0$ and $x_{21} < 20$), the first-order conditions are:

$$\frac{1}{2}\sqrt{\frac{x_{21}}{x_{11}}} - \frac{1}{2}\lambda\sqrt{\frac{20-x_{21}}{100-x_{11}}} = 0 \qquad \text{and} \qquad \frac{1}{2}\sqrt{\frac{x_{11}}{x_{21}}} - \frac{1}{2}\lambda\sqrt{\frac{100-x_{11}}{20-x_{21}}} = 0$$

Taking ratios, and solving for x_{21} in terms of x_{11} we find that $x_{21} = \frac{1}{5}x_{11}$. The Pareto set is then:

$$\mathcal{P} = \left\{ (x_{11}, x_{21}, x_{12}) : x_{21} = \frac{1}{5}x_{11} \text{ and } x_{12} = 100 - x_{11} \text{ for } x_{11} \in [0,100] \right\}$$

Graphically, this is exactly the diagonal in the Edgeworth box with money on the horizontal axis and number of cigarettes smoked on the vertical axis.

(iv) Absent the offer, consumer 1 will consume their entire endowment so will get utility $\sqrt{20 \times 50} = \sqrt{1000}$. Consumer 2 will offer T to consumer 1 to consume only x_{21} cigarettes according to:

$$\max_{(x_{12},x_{21})\geq \mathbf{0}, T\in\mathbb{R}} \sqrt{x_{12}\left(20-x_{21}\right)} \qquad \text{subject to:} \qquad x_{12}+T \leq 50$$
$$\text{and} \qquad \sqrt{\left(50+T\right)x_{21}} \geq \sqrt{1000}$$

Both constraints will bind at the optimum (no money will be left over since utility is increasing in money for both and Consumer 2 will offer T to exactly satisfy Consumer 1's participation constraint). Writing the second constraint with equality as $x_{21} = 1000/\left(50+T\right)$ and substituting:

$$\max_{T\in\mathbb{R}} \sqrt{\left(50-T\right)\left(20-\frac{1000}{50+T}\right)}$$

Squaring the objective function, expanding, dividing across by 20 and dropping constants, the problem becomes:

$$\max_{T\in\mathbb{R}} -\frac{2500}{50+T} - T + \frac{50T}{50+T}$$

This has a first-order condition:

$$\begin{aligned}
\frac{2500}{\left(50+T\right)^2} - 1 + \frac{50\left(50+T\right)-50T}{\left(50+T\right)^2} &= 0 \\
\frac{2500}{\left(50+T\right)^2} - 1 + \frac{2500}{\left(50+T\right)^2} &= 0 \\
5000 &= \left(50+T\right)^2 \\
5000 &= 2500 + 100T + T^2 \\
T^2 + 100T - 2500 &= 0
\end{aligned}$$

The positive root is:

$$\frac{-100+\sqrt{20000}}{2} = \frac{-100+\sqrt{2}\sqrt{10000}}{2} = \frac{-100+100\sqrt{2}}{2} = 50\left(\sqrt{2}-1\right) \approx 20.71$$

The cigarette consumption is then: $x_{21} = \frac{1000}{50+50\left(\sqrt{2}-1\right)} = \frac{20}{\sqrt{2}} \approx 14.14$

Question 4

Consider an economy with 2 goods, a consumption good x and a composite commodity m. There are 3 consumers with the following utility functions overs these 2 goods:

$$\begin{aligned}
u_1\left(m_1, x_1\right) &= m_1 + 4\log\left(x_1\right) \\
u_2\left(m_2, x_2\right) &= m_2 + 8\log\left(x_2\right) \\
u_3\left(m_3, x_3\right) &= m_3 + 12\log\left(x_3\right)
\end{aligned}$$

where $\log(\cdot)$ is the natural logarithm. Consumers can only consume a nonnegative quantity of the good x. There are two firms that produce the consumption good (output q) from the composite commodity (input z). Their production sets are:

$$Y_1 = \left\{(-z_1, q_1) : q_1 \geq 0 \text{ and } z_1 \geq \frac{1}{2}q_1^2\right\}$$
$$Y_2 = \left\{(-z_2, q_2) : q_2 \geq 0 \text{ and } z_2 \geq q_2^2\right\}$$

The total endowment of the composite commodity in the economy is $\bar{\omega}_m = 24$. There is no initial endowment of the consumption good.

(i) Formally specify the social planner's problem assuming the planner maximizes the unweighted sum of the consumers' utilities. Solve for the Pareto optimal allocation of x_i across consumers and q_j across firms. *Solution:*
The social planner solves the following problem:

$$\max_{\{(m_1,m_2,m_3,x_1,x_2,x_3,z_1,z_2,q_1,q_2)\}} m_1 + 4\log(x_1) + m_2 + 8\log(x_2) + m_3 + 12\log(x_3)$$

subject to:

- The technology constraints: $(-z_j, q_j) \in Y_j$ for $j = 1, 2$.
- The feasibility constraints:
  - $\sum_{i=1}^3 m_i = 24 - \sum_{j=1}^2 z_j$
  - $\sum_{i=1}^3 x_i = \sum_{j=1}^2 q_j$

Because utility is increasing in both goods, the firms will operate efficiently so that $z_1 = \frac{1}{2}q_1^2$ and $z_2 = q_2^2$. Using this in the first feasibility constraint and substituting this constraint in the objective for $\sum_{i=1}^3 m_i$ gives:

$$\max_{\{(x_1,x_2,x_3,q_1,q_2)\}} 4\log(x_1) + 8\log(x_2) + 12\log(x_3) + \bar{\omega}_m - \frac{1}{2}q_1^2 - q_2^2$$

subject to: $\sum_{i=1}^3 x_i = \sum_{j=1}^2 q_j$. Letting λ be the Lagrange multiplier on the constraint we get the following 5 first-order conditions:

- $\frac{4}{x_1} = \lambda$.
- $\frac{8}{x_2} = \lambda$.
- $\frac{12}{x_3} = \lambda$.
- $q_1 = \lambda$.

- $2q_2 = \lambda$.

Using these together in the constraint gives:

$$\frac{4}{\lambda} + \frac{8}{\lambda} + \frac{12}{\lambda} = \lambda + \frac{1}{2}\lambda$$

Solving for λ gives $\lambda = 4$. Using this value in the first-order conditions gives $(x_1, x_2, x_3) = (1, 2, 3)$ and $(q_1, q_2) = (4, 2)$.

(ii) Use (i) to specify the Pareto frontier. *Solution:*
Using the answer in (i), the total cost to produce the consumption good is $z_1 + z_2 = \frac{1}{2}q_1^2 + q_2^2 = 12$. This means that $\sum_{i=1}^{3} m_i = \bar{\omega}_m - \sum_{j=1}^{2} z_j = 24 - 12 = 12$ is available to distribute to consumers. The Pareto frontier is then:

$$\sum_{i=1}^{3} m_i + 4\log(1) + 8\log(2) + 12\log(3)$$

subject to $\sum_{i=1}^{3} m_i = 12$. The Pareto frontier is a plane in 3-dimensional Euclidean space.

(iii) Assume the price of the consumption good is p and the price of the composite commodity is normalized to 1. The distribution of the initial endowment of the composite commodity is $(\omega_{m1}, \omega_{m2}, \omega_{m3}) = (8, 8, 8)$. There is no initial endowment of the consumption good. Each consumer owns an equal share in the profits of each firm. Solve for the equilibrium price p and allocation of both goods (for both consumers and firms). *Solution:*
Given price p, firm 1's problem is:

$$\max_{q_1 \geq 0} pq_1 - \frac{1}{2}q_1^2$$

In an interior solution, the first-order condition is $q_1 = p$, which gives firm 1's supply function as $q_1(p) = p$. Similarly, for firm 2 we can get $q_2(p) = \frac{1}{2}p$. Firm profits are:

$$\pi_1(p) = pq_1(p) - \frac{1}{2}[q_1(p)]^2 = \frac{1}{2}p^2$$
$$\pi_2(p) = pq_2(p) - [q_2(p)]^2 = \frac{1}{4}p^2$$

Each consumer therefore gets a share $\frac{1}{3}\left[\frac{1}{2}p^2 + \frac{1}{4}p^2\right] = \frac{p^2}{4}$ in their budget constraint, on top of the value of the endowment ω_{mi}. Consumer 1's problem is:

$$\max_{(m_1,x_1)} m_1 + 4\log\left(x_1\right)$$

subject to $px_1 + m_1 \leq \omega_{m1} + \frac{p^2}{4}$ and $x_1 \geq 0$. Because utility is increasing in both goods, the budget constraint will bind and we can substitute it into the objective:

$$\max_{x_1 \geq 0} 8 + \frac{p^2}{4} + 4\log\left(x_1\right) - px_1$$

In an interior solution we have a first-order condition $\frac{4}{x_1} = p$. This gives a demand function $x_1\left(p\right) = \frac{4}{p}$. Repeating for consumers 2 and 3 we get $x_2\left(p\right) = \frac{8}{p}$ and $x_3\left(p\right) = \frac{12}{p}$. In equilibrium we have market clearing and demand for good x equals supply:

$$\underbrace{\frac{4}{p} + \frac{8}{p} + \frac{12}{p}}_{\text{Aggregate demand}} = \underbrace{p + \frac{1}{2}p}_{\text{Aggregate supply}}$$

Solving for p yields $p = 4$. At $p = 4$ consumer 1's consumption of the composite commodity is:

$$m_1 = \omega_{m1} + \frac{p^2}{4} - px_1 = 8 + \frac{4^2}{4} - 4 \times 1 = 8.$$

Similarly for consumers 2 and 3 we get $m_2 = 4$ and $m_3 = 0$.

The full allocation is then:

- $(x_1, x_2, x_3) = (1, 2, 3)$.
- $(m_1, m_2, m_3) = (8, 4, 0)$.
- $(q_1, q_2) = (4, 2)$.
- $(z_1, z_2) = (8, 4)$.

(iv) Show that your answer to (iii) is on the Pareto frontier in (ii). *Solution:*
The consumers and firms consume/produce the same quantities of the consumption good. The total consumption of m_i is 12, which satisfies the condition $\sum_{i=1}^{3} m_i = 12$ on the frontier. Thus the competitive equilibrium is a point on the frontier.